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Decoherence and Correlations in Semiclassical Cosmology

M. Castagnino[1] and S. Landau[1]

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Using the mathematical structure of rigged Hilbert spaces, for both in and out Fock spaces of a simple cosmological model, we verify the two requirements for classicality: decoherence and correlations.

1. INTRODUCTION

We know that the laws of classical mechanics describe with a high degree of accuracy the behavior of macroscopic systems, and yet it is believed that the phenomena on all scales and including the entire universe follow the laws of quantum mechanics. So, if we want to reconcile the two last statements, it is essential to understand the transition from the quantum to the classical regime. One of the scenarios where this problem is relevant is quantum cosmology, in which one attempts to apply quantum mechanics to closed cosmologies. This involves a problem that has not been solved; namely, quantizing the gravitational field. Therefore, it is an important issue to predict the conditions under which the gravitational field may be regarded as classical.

The point of view we will take is that there are two requirements that a system must fulfill to be regarded as classical: decoherence and correlations.

The first requirement is that the state that describes the system must be approximately classical, that is, that the off-diagonal terms of the reduced density matrix may be exceedingly smaller than the diagonal terms.

The second requirement is that evolution to a very good approximation should be described by classical laws. This means that the wave function or some distribution constructed from it (for example, the Wigner function

[1] Instituto de Astronomía y Física del Espacio, 1428 Buenos Aires, Argentina.

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criterion [9]) should be strongly peaked about a classical configuration, i.e., a configuration where coordinates and momenta are strongly correlated.

In a previous work [1], one of us studied the problem of choosing the mathematical structure that better explains the time asymmetry of nature. From this work, we know that if we want to retain the time-symmetric laws of nature and at the same time explain the time asymmetry of the universe, we must choose a space of solutions which is not time-symmetric.

In the context of semiclassical cosmology from a fully quantized cosmological model, a variable a and a time parameter η can be defined so that the function $a = a(\eta)$ can be found. When $\eta \to \infty$ we obtain a classical geometry $g_{\mu\nu}^{\text{out}}$ for the universe. Using the time η, we can transform the Wheeler–DeWitt equation into a Schrödinger equation with the corresponding Hamiltonian $h(a_{\text{out}})$. Using $h(a_{\text{out}})$ and the classical geometry $g_{\mu\nu}^{\text{out}}$, we can find a semiclassical vacuum state $|0, \text{out}\rangle$ which diagonalizes the Hamiltonian, the creation and annihilation operators related to this vacuum, and the corresponding Fock spaces.

As we have already explained, to obtain a time-asymmetric evolution, we will choose time-asymmetric Fock spaces. A convenient choices of time-asymmetric spaces was proposed in ref. 1.

2. THE MODEL

We now introduce the model we will use to illustrate the method proposed in ref. 1. Let us consider a Robertson–Walker metric (we will mostly consider the flat case)

$$dS = a^2\,(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{1}$$

where η is the conformal time and $a(\eta)$ the Robertson–Walker scale factor. The total action is given by

$$S = S_g + S_f + S_i \tag{2}$$

where S_g is the gravitational action, S_f is the usual action of a spinless massive scalar field φ, conformally coupled ($\xi = 1/6$), and S_i is the interaction given by a mass term in Robertson–Walker geometry. The gravitational action is given by

$$S_g = M^2 \int d\eta \left[-\frac{1}{2}\dot{a}^2 - V(a) \right] \tag{3}$$

where M is the Planck mass, $\dot{a} = da/d\eta$, and $V(a)$ is the potential function that arises from a spatial curvature, a possible cosmological constant, and, eventually, a classical matter field.

In this paper we will consider the potential function used by Birrell and Davies [2] to illustrate the use of the adiabatic approximation in an asymptotically nonstatic four-dimensional cosmological model:

$$V(a) = \frac{B^2}{2}\left(1 - \frac{A^2}{a^2}\right) \tag{4}$$

where A and B are arbitrary constants.

The Wheeler–DeWitt equation for this model is

$$H\Psi(a, \varphi) = (h_g + h_f + h_i)\Psi(a, \varphi) = 0 \tag{5}$$

where

$$h_g = \frac{1}{2M}\,\partial_a^2 + M^2 V(a) \tag{6}$$

$$h_f = -\frac{1}{2}\int_k (\partial_{\varphi k}^2 - k^2\varphi_k^2)\,dk \tag{7}$$

$$h_i = \frac{m^2 a^2}{2}\int_k \varphi_k^2\,dk \tag{8}$$

and m is the mass of the scalar field.

To analyze decoherence and correlations, we need to define a time parameter; so we will work in the semiclassical regime. Therefore let us go to the semiclassical case using the WKB ansatz, namely

$$\Psi(a, \varphi) = \chi(a, \varphi)\exp[iM^2 S(a)] \tag{9}$$

Let us now assume that $S(a)$ and $\chi(a, \varphi)$ can be expanded in powers of M^{-1}:

$$S = S_0 + M^{-1}S_1 + M^{-2}S_2 + \dots \tag{10}$$

$$\chi = \chi_0 + M^{-1}\chi_1 + M^{-2}\chi_2 + \dots \tag{11}$$

If we substitute the last equations into Eq. (5), we obtain a set a set of equations involving all the functions appearing in the expansion. To leading order (i.e., M^2), we get

$$\left[\frac{dS(a)}{da}\right]^2 = 2V(a) \tag{12}$$

which is essentially the Hamilton–Jacobi equation for the variable $a(\eta)$. From this equation we can deduce $S(a)$.

Then we can define the time parameter in our model as follows: first we write the momentum associated to $a(\eta)$, namely

$$\frac{da}{dt} = \frac{dS}{da}$$

Then, the (semi)classical time parameter η is given by

$$\frac{d}{d\eta} = \frac{dS}{da}\frac{d}{da} = \pm\sqrt{2V(a)}\,\frac{d}{da} \tag{13}$$

From Eqs. (12) and (13), we can find a set of classical solutions

$$a(\eta) = \pm\,(A^2 + B^2\eta^2)^{1/2} + C \tag{14}$$

where C is a constant.

Using different values of the constant C and different choices of the signs $\pm$ we obtain different (semi)classical geometries. We will work only with two different cases: we will always choose $C = 0$ and we will study the cases with the plus and the minus sign. We will label these (semi)classical geometries with indexes α and β, respectively. From Eq. (14) we can see that in the asymptotic regions ($\eta \to \pm\infty$) the model approaches the radiation-dominated Friedmann cosmology.

Taking the following order in the WDW equation, we obtain a Schrödinger equation for $\chi(a, \varphi)$:

$$i\frac{d}{d\eta}\chi(a, \varphi) = -\frac{1}{2}\int_k [\partial^2{}_k - \Omega_k^2\varphi_k^2]\; dk\; \chi(a,\varphi) \tag{15}$$

where $\Omega_k^2 = m^2a^2 + k^2$.

As the coupling is conformal we will have well-defined vacua [2]. So, we consider now two scales a_{in} and a_{out} such that $0 < a_{\text{in}} << a_{\text{out}}$. Next, we define the corresponding $|0, \text{in}\rangle$, $|0, \text{out}\rangle$ vacua there, where $|0, \text{in}\rangle$ is the adiabatic vacuum for $\eta \to -\infty$ and $|0, \text{out}\rangle$ is the corresponding one for $\eta \to +\infty$. It is well known [2, 3] that we can diagonalize the time-dependent Hamiltonian (15) at a_{in} and at a_{out} and define the corresponding creation and annihilation operators, and the corresponding Fock spaces.

Thus, following Eqs. (37)–(43) of ref. 1, we can construct the Fock space and find the eigenvector of $h(a_{\text{out}})$ as follows:

$$\begin{aligned} h(a_{\text{out}})|\{k\}, \text{out}\rangle &= h(a_{\text{out}})|\varpi, [k], \text{out}\rangle \\ &= \Omega(a_{\text{out}})|\{k\}, \text{out}\rangle = \sum_{k\in\{k\}} \Omega_\varpi(a_{\text{out}})|\varpi, [k], \text{out}\rangle \end{aligned} \tag{16}$$

where $\varpi = k^2$, $[k]$ is the remaining set of labels necessary to define the vector unambiguously, and $|\varpi, [k], \text{out}\rangle$ is an orthonormal basis [1].

In the same way we can find the eigenvectors of $h(a_{\text{in}})$. Thus we can also define the S-matrix between the in and out states [Eq. (44), ref. 1]:

$$S_{\varpi,[k];\varpi',[k']} = \langle\varpi, [k], \text{in}|\varpi', [k'], \text{out}\rangle = S_{\varpi,[k];[k']}\,\delta(\varpi - \varpi') \tag{17}$$

Now we will define the Fock in and out spaces. We make the following choice: for the in Fock space we will use functions $|\psi\rangle \in \phi_{+,\text{in}}$, namely, such that $\langle\varpi, \text{in}|\psi\rangle \in S|_{R+}$ and $\langle\varpi, \text{in}|\psi\rangle \in H^2_+|_{R+}$, s is Schwartz space and where H^2_+ is the space of Hardy class function from above; and for the out Fock space we will use functions $|\varphi\rangle \in \phi_{-,\text{out}}$ such that $\langle\varpi, \text{out}|\varphi\rangle \in S|_{R+}$ and $\langle\varpi, \text{out}|\varphi\rangle \in H^2_-|_{R+}$. So we can obtain [1] a spectral decomposition for the $h(a_{\text{out}})$ (in a weak sense):

$$h(a_{\text{out}}) = \sum_n \Omega_n|\bar{n}\rangle\langle\bar{n}| + \int dz\, \Omega_z|\bar{z}\rangle\langle\bar{z}| \tag{18}$$

where $\Omega_n = m^2a^2_{\text{out}} + z_n$, and z_n are the poles of the S-matrix.

From refs. 4 and 6 it can be seen that S-matrix corresponding to this model has infinite poles and that the k corresponding to each pole reads

$$k^2 = mB\left[-\frac{mA^2}{B} - 2i\left(n + \frac{1}{2}\right)\right] \tag{19}$$

Thus we can compute the squared energy of each pole:

$$\Omega^2_n = m^2a^2 + mB\left[-\frac{mA^2}{B} - 2i\left(n + \frac{1}{2}\right)\right] \tag{20}$$

3. DECOHERENCE AND CORRELATIONS

In this section we show how classical behavior, namely decoherence and correlations, arises using the spectral decomposition of Eq. (18).

Decoherence is a dissipative process. It has been studied [12] as closely related to another dissipative process, namely, particle creation from the gravitational field during universe expansion. We will obtain as in ref. 5 a set of discrete unstable states, namely, unstable particles, and a set of continuous stable states [see Eq. (18)], the latter corresponding to the stable particles.

As the modes do not interact between themselves, we can write

$$\chi(a, \varphi) = \prod_{n=1}^{\infty} \chi_n(\eta, \varphi_n) \tag{21}$$

Thus we obtain a Schrödinger equation for each mode, namely

$$i\frac{d}{d\eta}\chi_n(a, \varphi_n) = -\frac{1}{2}[\partial^2_n - \Omega^2_n\varphi^2_n]\chi_n(a, \varphi_n) \tag{22}$$

Let now us now assume the Gaussian ansatz for $\chi_n(\eta, \varphi_n)$:

$$\chi_n(\eta, \varphi_n) = A_n(\eta) \exp[i\alpha_n(\eta) - B_n(\eta)\varphi_n^2] \tag{23}$$

where $A_n(\eta)$ and $\alpha_n(\eta)$ are real, while $B_n(\eta)$ may be complex, namely, $B_n(\eta) = B_{nR}(\eta) + iB_{ni}(\eta)$.

Solving Eq. (22) with the Gaussian ansatz, we obtain $\alpha_n(\eta)$ and $B_n(\eta)$:

$$\dot{\alpha}_n(\eta) = -B_{nR}(\eta) \tag{24}$$

$$\dot{B}_n(\eta) = -2iB_n^2(\eta) + \frac{i}{2}\Omega_n^2(\eta) \tag{25}$$

From the normalization of $\chi_n(\eta, \varphi_n)$ we obtain

$$A_n(\eta) = \pi^{-1/4}(2B_{nR}(\eta))^{1/4} \tag{26}$$

From these equations and working with the continuous spectrum [i.e., with the second part of (18)] it can be proved that there is loss of coherence [6, 7] only for certain initial conditions. Here we will see that, working with the discrete poles spectrum, as proposed in ref. 1, we will find decoherence for almost every initial condition.

Integrating the modes of the scalar field, we obtain the following reduced density matrix ρ_r:

$$\begin{aligned}\rho_r(a, a') = {} & \exp[-iMS_\alpha(a) + iMS_\alpha(a')]\rho_r^{\alpha\alpha}(a, a') \\ & + \exp[-iMS_\alpha(a) + iMS_\beta(a')]\rho_r^{\alpha\beta}(a, a') \\ & + \exp[-iMS_\beta(a) + iMS_\alpha(a')]\rho_r^{\beta\alpha}(a, a') \\ & + \exp[-iMS_\beta(a) + iMS_\beta(a')]\rho_r^{\beta\beta}(a, a')\end{aligned}$$

where

$$\rho_r^{\alpha\beta}(a, a') = \prod_{n=1}^{\infty} \rho_{rn}^{\alpha\beta}(a, a') = \prod_{n=1}^{\infty} \int d\varphi_n\, \chi_n^{\alpha}(\eta, \varphi_n)\chi_n^{\beta}(\eta, \varphi_n) \tag{27}$$

and α and β label the two different classical geometries.

It is convenient to introduce the following change of variable in order to characterize the wave function of each mode:

$$B_n = -\frac{1}{2}\frac{\dot{g}_n}{g_n} \tag{28}$$

Let us now consider a solution the second-order adiabatic expansion of function g_n in the limit $\eta \to +\infty$:

$$g_n = \frac{P_n}{\sqrt{2\Omega_n}} \exp\left[-i \int \Omega_n(\eta')\, d\eta' \right] + \frac{Q_n}{\sqrt{2\Omega_n}} \exp\left[i \int \Omega_n(\eta')\, d\eta' \right] \tag{29}$$

where P_n and Q_n are arbitrary coefficients. It is obvious that if all the Ω_n are real, the last equation will have an oscillatory nature, as will its derivatives. This will also be the behavior of B_n. Then we see that when $\eta \to +\infty$, B_n is an oscillatory function with no limits in general. We will only have good limits for some particular cases [6, 8, 9]. But if Ω_n is complex, because $\text{Im}\,\Omega_n \neq 0$, a damping and a growing factor appear in Eq. (29), and B_n reads in the limit $\eta \to \infty$

$$B_n \approx \tfrac{1}{2}\Omega_n \tag{30}$$

From Eqs. (14), (20), and (30) we can compute the expression for B_n for both semiclassical solutions α and β:

$$B_n(\eta, \alpha) = B_n(\eta, \beta) = \frac{\sqrt{2}}{4}\left[m^2B^2\eta^2 + \left(m^4B^4\eta^4 + 4m^2B^2\left(n + \frac{1}{2}\right)^2\right)^{1/2}\right]^{1/2} - i\,\frac{\frac{1}{2}\sqrt{2}\, mB(n + \frac{1}{2})}{[m^2B^2\eta^2 + (m^4B^4\eta^4 + 4m^2B^2(n + \frac{1}{2})^2)^{1/2}]^{1/2}} \tag{31}$$

Now we will see, making the exact calculations, that in the limit $\eta \to \infty$ there is necessarily decoherence (a) for the same classical geometry if the times η and η' are different, namely $\rho_r^{\alpha\alpha}(\eta, \eta') \to 0$ and $\rho_r^{\beta\beta}(\eta, \eta') \to 0$ when $\eta \to \infty$, and (b) for different classical geometries ($\alpha \neq \beta$), i.e., $\rho_r^{\alpha\beta}(\eta, \eta') \to 0$ when $\eta \to \infty$.

From Eqs. (23) and (27) we obtain

$$\rho_{rn}^{\alpha\beta}(\eta, \eta') = \left(\frac{4B_{nR}(\eta, \alpha)B_{nR}(\eta', \beta)}{[B_n^*(\eta, \alpha) + B_n(\eta', \beta)]^2} \right)^{1/4} \exp[-i\alpha_n(\eta, \alpha) + i\alpha_n(\eta', \beta)] \tag{32}$$

(**a**) Using Eqs. (30) and (31), we calculate the asymptotic behavior ($\eta, \eta' \to \infty$) of $\rho_{rn}^{\alpha\alpha}(\eta, \eta')$, namely

$$|\rho_{rn}^{\alpha\alpha}(\eta, \eta')| \cong \left[\frac{4\eta\eta'}{[\eta + \eta']^2} \right]^{1/4} \tag{33}$$

Making the change of variables $(\eta - \eta')/2 = \Delta$, $(\eta + \eta')/2 = \bar{\eta}$ with $\Delta \ll 1$, we obtain

$$|\rho_{rn}^{\alpha\alpha}(\eta, \eta')| \cong \left[1 - \left(\frac{\Delta}{\eta}\right)^2\right]^{1/4} \tag{34}$$

Since $|\rho_{rn}^{\alpha\alpha}(\eta, \eta')| \leq 1$ with the equality only if $\eta = \eta'$, it is easy to see that $\rho_r^{\alpha\alpha}(\eta, \eta')$ is equal to zero if $\eta \neq \eta'$. This means that the reduced density matrix has diagonalized perfectly, i.e., we have achieved perfect decoherence. However, it is known [6, 7] that perfect decoherence also implies that the Wigner function has an infinite spread, so we cannot say that the system is classical.

On the other hand, Dowker and Kent [10, 11], working with the consistent histories formalism, made the assumption that exactly consistent sets of histories must be found very close to an approximately consistent set. Thus, having found the exact consistent set of histories, it would be reasonable to say that there are many approximate consistent sets near it. Although we are not working with this formalism, we can consider geometries α and β to be different histories. Since having an exact consistent set of histories means in our formalism exact decoherence, we can try to find the approximate consistent sets near the exact one, in order to obtain both requirements for classical behavior, namely decoherence and correlations.

If we introduce a cutoff in Eq. (27) at some very large value of n, $n = N$, the reduced density matrix is no longer exactly diagonal, i.e., we obtain the "approximate" consistent set. We will postpone for a future work the discussion about the value of N. However, it would be useful to consider what conditions has N to fulfill in order to achieve decoherence and correlations.

Thus we obtain if $\eta \approx \eta'$

$$|\rho_r^{\alpha\alpha}(\eta, \eta')| = \left|\prod_{n=1}^{N} \rho_{rn}^{\alpha\alpha}(\eta, \eta')\right| \approx \exp\left[-\frac{N}{4}\left(\frac{\Delta}{\eta}\right)^2\right] \tag{35}$$

From the last equation, we observe that the reduced density matrix turns out to be a Gaussian of width σ_d, where

$$\sigma_d = \frac{2\eta}{N^{1/2}} \tag{36}$$

Thus, one must have $\sqrt{N} \gg 1$ in order to obtain decoherence.

(**b**) From Eqs. (31) and (32) we compute $\rho_r^{\beta\beta}(\eta, \eta')$ and $\rho_r^{\alpha\beta}(\eta, \eta')$ and obtain for $\eta \to \infty$ as in Eq. (33)

$$|\rho_{rn}^{\beta\beta}(\eta, \eta')| = |\rho_{rn}^{\alpha\beta}(\eta, \eta')| \cong \left[\frac{4\eta\eta'}{[\eta + \eta']^2}\right]^{1/4} \tag{37}$$

So, following the same steps as we did for $\rho_{rn}^{\alpha\alpha}(\eta, \eta')$, (33)–(36), we obtain decoherence for different conformal times and for different classical geometries. It is easy to see that we can follow the same steps for $\rho_{rn}^{\alpha\beta}(\eta, \eta')$ since from Eq. (31), $B_n(\eta, \alpha) = B_n(\eta, \beta)$.

Next we will analyze the existence of correlations between coordinates and momenta using the Wigner function criterion [9]. Since correlations between coordinates and momenta should be examined "inside" each classical branch, we compute the Wigner function associated with each (semi)classical solution. The Wigner function associated with the reduced density matrix given by Eqs. (27) and (32) is [7]

$$F_W^{\alpha\alpha}(a, P) \cong C^2(\eta)\sqrt{\frac{\pi}{\sigma_c^2}}\exp\left[-\frac{(P - M^2\dot{S} + \Sigma_{n=1}^N(\dot{\alpha}_n - \dot{B}_{ni}/4B_{nR}))^2}{\sigma_c^2}\right] \tag{38}$$

where

$$\sigma_c^2 = \sum_{n=1}^N \frac{|\dot{B}_n|^2}{4B_{nR}^2} \tag{39}$$

We can predict strong correlation when the center of the peak of the Wigner function is large compared to the spread, i.e., when

$$\left(M^2\dot{S} - \sum_{n=1}^N\left(\dot{\alpha}_n - \frac{\dot{B}_{ni}}{4B_{nR}}\right)\right)^2 \gg \sigma_c^2 \tag{40}$$

Using the same approximation we made for calculating the reduced density matrix, we obtain the following expression for the width of Wigner function:

$$\sigma_c^2(\eta, \alpha) \cong \frac{N}{4\eta^2} \tag{41}$$

We can see that σ_c is the inverse of σ_d, (35), confirming what we said following Eq. (34).

We also calculate the center of the peak of the Wigner function, namely:

$$\left(M^2\dot{S} - \sum_{n=1}^N\left(\dot{\alpha}_n - \frac{\dot{B}_{ni}}{4B_{nR}}\right)\right)^2 \cong m^2B^2N^2\eta^2 \tag{42}$$

From Eqs. (41) and (42) we observe the behavior of the center of the peak and the width of the Wigner function in the limit $\eta \to \infty$. Thus the condition for the existence of correlations turns out to be

$$N \gg \frac{1}{m^2 B^2 \eta^4} \tag{43}$$

So, if the value of the cutoff is such that $N \gg 1$ and $N \gg 1/m^2B^2\eta^4$, we can say that the system behaves classically: the off-diagonal terms of the reduced density matrix are exponentially smaller that the diagonal terms, while we can predict strong correlations between $a(\eta)$ and its conjugate momenta.

4. CONCLUSION

In the previous section we analyzed the existence of decoherence and correlations within the model proposed in Section 2. In order to obtain decoherence and correlations we introduced a cutoff N and found restrictions on its value: $N \gg 1$ and $N \gg 1/m^2B^2\eta^4$. This choice seems to be natural, since really we are working in the limit $\eta \to \infty$.

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